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Lie Algebra Bases for the Pseudo-Orthogonal Groups

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Abstract

The isomorphic mappings between the canonical bases appearing in the general structure theory of semi-simple real Lie algebras and the bases obtained directly from the pseudo-orthogonal groups are investigated in detail, and it is shown that these mappings can be cast in a remarkable simple form which is valid for all cases.

1. *Introduction*

The pseudo-orthogonal groups $SO(p, q)$ are of considerable interest in theoretical physics. The most important are $SO(3, 1)$ (the homogeneous Lorentz group), $SO(3, 2)$ and $SO(4, 1)$ (the de Sitter groups), and $SO(4, 2)$ (the conformal group), but others, such as $SO(2, 1)$, have also found many applications.

For many purposes it is convenient to work directly with the definition of $SO(p, q)$ as the set of real $(p + q) \times (p + q)$ matrices $\mathbf{u}$ such that $\det \mathbf{u} = +1$ and

$$\tilde{\mathbf{u}}\mathbf{g}\mathbf{u} = \mathbf{g}$$

where the tilde denotes the transpose and $\mathbf{g}$ is a diagonal matrix with diagonal elements ± 1, p being of one sign and q of the other. The Lie algebra $so(p, q)$ of $SO(p, q)$ may then be defined as the set of real $(p + q) \times (p + q)$ matrices $\mathbf{a}$ such that $\operatorname{tr} \mathbf{a} = 0$ and

$$\tilde{\mathbf{a}}\mathbf{g} + \mathbf{g}\mathbf{a} = \mathbf{0} \tag{1.1}$$

On the other hand it is often more convenient to work with the canonical form $\mathscr{L}$ of the Lie algebra $so(p, q)$ that appears naturally in the standard structure theory of real semi-simple Lie algebras. (This theory, due mainly to Cartan (1914, 1929) and Gantmacher (1939a, 1939b), will be summarized briefly in Section 2.)

Even in two closely related problems it can happen that the $so(p, q)$ basis is the most useful for one problem while the canonical basis is the most convenient for the other. For example, the recent determination of the maximal solvable subgroups of $SO(p, q)$ by Patera, Winternitz & Zassenhaus (1974) employs the $so(p, q)$ basis, while for embeddings of $SO(p, q)$ with other semi-simple Lie groups (Cornwell, 1971a, 1971b, 1972; Ekins & Cornwell, 1974a, 1974b, 1974c) the canonical basis is the most convenient.

Similarly in considering the infinite-dimensional unitary irreducible representations of the non-compact groups $SO(p, q)$ the canonical form has been used by Joseph (1970) and Joseph & Hieggelke (1970), while Fischer *et al.* (1966), Limic *et al.* (1966), and Raczka *et al.* (1966) found the $so(p, q)$ basis the more convenient.

It would therefore be useful to have a simple expression for the isomorphic mapping between the two sets of bases. Of course it is obvious (at least in outline) how this can be achieved, that is, by choosing a matrix representation of dimension $(p + q)$ of the corresponding complex Lie algebra and then applying an appropriate similarity transformation. The purpose of this note is to demonstrate the rather surprising fact that, with a judicious choice of conventions, this similarity transformation can be cast in a particularly neat and simple form, which is given explicitly for *all* cases.

2. *The Canonical Basis*

In this section a brief account will be given of the construction of the canonical basis of $\mathscr{L}$ along the lines developed by Cartan (1929) and Gantmacher (1939a, 1939b). (Further details using the same conventions as in the present note may be found in the papers of Cornwell (1971a, 1971b, 1972).)

For $p + q$ odd, $so(p, q)$ is a real Lie algebra whose complexification $\tilde{\mathscr{L}}$ is B_l, where $p + q = 2l + 1$. For $p + q$ even, $\tilde{\mathscr{L}}$ is D_l, where $p + q = 2l$. (Only the case $l \geqslant 2$ will be considered, as when $p + q = 2$ the Lie group $SO(p, q)$ is Abelian.) Each complex Lie algebra $\tilde{\mathscr{L}}$ has a compact real Lie algebra $\mathscr{L}_c$ (which is isomorphic to $so(p + q)$). The construction of the real non-compact semi-simple Lie algebras $\mathscr{L}$ having complexification $\tilde{\mathscr{L}}$ can be accomplished by the following theorem of Cartan (1929): 'First find the involutive automorphisms S of $\mathscr{L}_c$. Then take a basis of $\mathscr{L}_c$ consisting of the "eigenvectors" of S, multiply those eigenvectors having eigenvalue -1 by i and leave the remaining eigenvectors unchanged. To the basis so obtained there corresponds a real form $\mathscr{L}$ of $\tilde{\mathscr{L}}$.'

As is well known, $\tilde{\mathscr{L}}$ itself has a canonical form, for which it is most convenient to use the conventions of Jacobson (1962) (particularly Chapter 4 and especially equation (28) of page 121). The basis elements of the Cartan subalgebra $\mathscr{H}$ of $\tilde{\mathscr{L}}$ will be denoted by $h_1, h_2, \ldots, h_l$, a general element of $\mathscr{H}$ will be denoted by h, α denotes a root and e_α the corresponding basis element of $\tilde{\mathscr{L}}$ such that $[e_\alpha, h] = \alpha(h)e_\alpha$. The canonical basis of the compact real form $\mathscr{L}_c$ of $\tilde{\mathscr{L}}$ may then be taken to consist of the elements ih_j, $j = 1, 2, \ldots, l$, and the elements $(e_\alpha + e_{-\alpha})$ and $i(e_\alpha - e_{-\alpha})$ for every root α of $\tilde{\mathscr{L}}$.

Gantmacher (1939a) has shown that *every* automorphism S of $\mathscr{L}_c$ can be written in the form $S = U^{-1}ZU$, where U is an inner automorphism of $\mathscr{L}_c$ and Z is a *chief* automorphism of $\mathscr{L}_c$. Clearly S is involutive if and only if Z is involutive. As S and Z then generate isomorphic real forms, it is sufficient to consider only the real forms generated by Z.

Gantmacher (1939a) has also shown that every chief inner automorphism of $\mathscr{L}_c$ has the form exp(ad h), where $h \in \mathscr{H} \cap \mathscr{L}_c$, and ad denotes the adjoint representation of $\mathscr{L}_c$. Each of the basis elements of the above canonical basis for $\mathscr{L}_c$ is then an eigenvector of Z, the elements ih_j corresponding to eigenvalue $+1, j = 1, 2, \ldots, l$, while $(e_\alpha + e_{-\alpha})$ and $i(e_\alpha - e_{-\alpha})$ both correspond to eigenvalue $\exp\{\alpha(h)\}$ $(=\pm 1)$.

The compact real form $\mathscr{L}_c$ of B_l has only inner automorphisms. The chief inner involutive automorphism $Z = \exp(\text{ad } h)$ may be chosen so that for the real form $\mathscr{L}$ isomorphic to $so(2l + 1 - 2r, 2r)$, $r = 1, \ldots, l$,

$$\exp\{\alpha_j(h)\} = \begin{cases} +1, & j = 1, \ldots, l-1; j \neq l-r \\ -1, & j = l-r, l \end{cases} \tag{2.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_l$ are the simple roots of B_l (cf. Cornwell, 1971a, Section 4.2).

For D_l the situation is more complicated, as $\mathscr{L}_c$ has both inner and outer automorphisms. The real form $\mathscr{L}$ isomorphic to $so(2l - 2r, 2r)$, $r = 1, \ldots, [\frac{1}{2}l]$, (where $[\frac{1}{2}l]$ denotes the largest integer $\leqslant \frac{1}{2}l$) is generated by the chief inner automorphism $Z = \exp(\text{ad } h)$, where h may be chosen so that

$$\exp\{\alpha_j(h)\} = \begin{cases} +1, & j = 1, \ldots, l;\ j \neq r \\ -1, & j = r \end{cases} \tag{2.2}$$

(cf. Cornwell, 1971a, Section 4.4). The real form $\mathscr{L}$ isomorphic to $so(2l - 2r - 1, 2r + 1)$, $r = 0, 1, \ldots, [\frac{1}{2}l]$, is generated by the chief outer automorphism $Z = Z_0 \exp(\text{ad } h)$, where h may be chosen so that for $r \geqslant 1$

$$\exp\{\alpha_j(h)\} = \begin{cases} +1, & j = 1, \ldots, l;\ j \neq l-r-1 \\ -1, & j = l-r-1 \end{cases} \tag{2.3}$$

while for $r = 0$

$$\exp\{\alpha_j(h)\} = \begin{cases} +1, & j = 1, \ldots, l-2 \\ -1, & j = l-1, l \end{cases} \tag{2.4}$$

(cf. Cornwell, 1972). Here Z_0 is an automorphism of $\mathscr{L}_c$ (and hence of $\mathscr{L}$) such that

$$\left.\begin{array}{llll} Z_0 h_j = h_j, & j = 1, \ldots, l-2; & Z_0 h_{l-1} = h_l; & Z_0 h_l = h_{l-1}; \\ Z_0 e_j = e_j, & j = 1, \ldots, l-2; & Z_0 e_{l-1} = e_l; & Z_0 e_l = e_{l-1}; \\ Z_0 e_{-j} = e_{-j}, & j = 1, \ldots, l-2; & Z_0 e_{-(l-1)} = e_{-l}; & Z_0 e_{-l} = e_{-(l-1)}; \end{array}\right\}$$

(cf. Gantmacher, 1939a, 1939b; Cornwell, 1972), where e_j, e_{-j} denote the basis elements of $\mathscr{L}$ corresponding to the roots α_j and $-\alpha_j$ respectively. This implies that ih_j, $(e_j + e_{-j})$, and $i(e_j - e_{-j})$, $j = 1, 2, \ldots, l-2$, together with

$i(h_{l-1}+h_l)$, $\{(e_{l-1}+e_{-(l-1)})+(e_l+e_{-l})\}$, and $i\{(e_{l-1}-e_{-(l-1)})+(e_l-e_{-l})\}$ are eigenvectors of Z_0 with eigenvalue +1, while $i(h_{l-1}-h_l)$, $\{(e_{l-1}+e_{-(l-1)})-(e_l+e_{-l})\}$, and $i\{(e_{l-1}-e_{-(l-1)})-(e_l-e_{-l})\}$ are eigenvectors of Z_0 with eigenvalue −1. For the non-simple roots α, $Z_0e_\alpha = \chi_\alpha e_\beta$, where $\chi_\alpha = \chi_{-\alpha} = \pm 1$ and the root β is related to α in a known way (cf. Gantmacher, 1939a). If $\alpha = \beta$, then $(e_\alpha + e_{-\alpha})$ and $i(e_\alpha - e_{-\alpha})$ are eigenvectors of Z_0 with eigenvalue +1, while if $\alpha \neq \beta$, $\{(e_\alpha+e_{-\alpha})+(e_\beta+e_{-\beta})\}$ and $i\{(e_\alpha-e_{-\alpha})+(e_\beta-e_{-\beta})\}$ are eigenvectors with eigenvalue +1, and $\{(e_\alpha+e_{-\alpha})-(e_\beta+e_{-\beta})\}$ and $i\{(e_\alpha-e_{-\alpha})-(e_\beta-e_{-\beta})\}$ are eigenvectors with eigenvalue −1. As the h of $Z = Z_0 \exp(\mathrm{ad}\, h)$ has the property that $\alpha(h) = \beta(h)$ when $Z_0e_\alpha = \chi_\alpha e_\beta$ (even when the roots α and β are not identical), it follows that all of the above eigenvectors of Z_0 are also eigenvectors of $Z = Z_0 \exp(\mathrm{ad}\, h)$, with eigenvalues that are easily deduced.

The complex Lie algebra B_l may be realised (cf. Konuma *et al.*, 1963, page 36) as the set of $(2l + 1)$-dimensional complex matrices $\mathbf{B}$ such that $\mathrm{tr}\,\mathbf{B} = 0$ and

$$\tilde{\mathbf{B}}\mathbf{G} + \mathbf{G}\mathbf{B} = \mathbf{0} \tag{2.5}$$

where

$$\mathbf{G} = \begin{pmatrix} \mathbf{I}(1) & 0 & 0 \\ 0 & \mathbf{0}(l) & \mathbf{I}(l) \\ 0 & \mathbf{I}(l) & \mathbf{0}(l) \end{pmatrix} \tag{2.6}$$

(Here $\mathbf{0}(l)$ and $\mathbf{I}(l)$ denote the $l \times l$ zero and identity matrices respectively.) An explicit $(2l + 1)$-dimensional matrix representation of the basis elements $h_1, h_2, \ldots, h_l$ of the Cartan subalgebra and of the basis vectors e_j corresponding to the simple roots of B_l is then given by

$$\begin{aligned} \mathbf{h}_j &= -\mathbf{e}_{j+1,j+1} + \mathbf{e}_{j+l+1,j+l+1} + \mathbf{e}_{j+2,j+2} - \mathbf{e}_{j+l+2,j+l+2}, \qquad j = 1, \ldots, l-1 \\ \mathbf{h}_l &= 2\{-\mathbf{e}_{l+1,l+1} + \mathbf{e}_{2l+1,2l+1}\} \\ \mathbf{e}_j &= \{\mathbf{e}_{j+1,j+2} - \mathbf{e}_{j+l+2,j+l+1}\}\{2(2l-1)\}^{-1/2}, \qquad j = 1, \ldots, l-1 \\ \mathbf{e}_l &= \{\mathbf{e}_{1,2l+1} - \mathbf{e}_{l+1,1}\}\{2(2l-1)\}^{-1/2} \end{aligned} \tag{2.7}$$

(Cornwell, 1971a, Appendix A). (Here $(\mathbf{e}_{jk})_{uv} = \delta_{ju}\delta_{kv}$.)

Similarly the complex Lie algebra D_l may be realised as the set of $2l$-dimensional complex matrices $\mathbf{B}$ satisfying $\mathrm{tr}\,\mathbf{B} = 0$ and condition (2.5), but now with

$$\mathbf{G} = \begin{pmatrix} \mathbf{0}(l) & \mathbf{I}(l) \\ \mathbf{I}(l) & \mathbf{0}(l) \end{pmatrix} \tag{2.8}$$

in place of (2.6). The corresponding $2l$-dimensional matrix representation of the generators $h_1, \ldots, h_l$ and $e_1, \ldots, e_l$ for D_l is then given by

$$\left.\begin{aligned} \mathbf{h}_j &= -\mathbf{e}_{jj} + \mathbf{e}_{j+l,j+l} + \mathbf{e}_{j+1,j+1} - \mathbf{e}_{j+l+1,j+l+1}, \qquad j = 1, \ldots, l-1 \\ \mathbf{h}_l &= -\mathbf{e}_{l-1,l-1} + \mathbf{e}_{2l-1,2l-1} - \mathbf{e}_{ll} + \mathbf{e}_{2l,2l} \\ \mathbf{e}_j &= \{\mathbf{e}_{j,j+1} - \mathbf{e}_{j+l+1,j+l}\}\{4(l-1)\}^{-1/2}, \qquad j = 1, \ldots, l-1 \\ \mathbf{e}_l &= \{\mathbf{e}_{l-1,2l} - \mathbf{e}_{l,2l-1}\}\{4(l-1)\}^{-1/2} \end{aligned}\right\} \quad (2.9)$$

(Cornwell, 1971a, Appendix A).

3. *Similarity Transformation from the Canonical Form to* $so(2l + 1 - 2r, 2r)$, $r = 0, 1, \ldots, l$

Theorem. Let **b** be an element of the matrix realisation of the canonical form $\mathscr{L}$. Then the similarity transformation to the $so(2l + 1 - 2r, 2r)$ Lie algebra (for $r = 0, 1, \ldots, l$) is given by

$$\mathbf{a} = \mathbf{S}\mathbf{b}\mathbf{S}^{-1}$$

where

$$\mathbf{S} = (\sqrt{\mathbf{g}})\mathbf{T} \qquad (3.1)$$

T being given by

$$(\mathbf{T})_{jk} = \begin{cases} 1, & j = 2k-2, \quad k = 2, \ldots, l+1, \quad \text{and} \quad j = 2k - 2l - 2, \\ & k = l+2, \ldots, 2l+1 \\ i, & j = 2k-3, \quad k = 2, \ldots, l+1 \\ -i, & j = 2k - 2l - 3, \quad k = l+2, \ldots, 2l+1 \\ \sqrt{2}, & j = 2l+1, \quad k = 1 \\ 0, & \text{all other } j, k, \end{cases} \qquad (3.2)$$

provided that the diagonal elements of **g** of (1.1) are arranged so that

$$g_{2j,2j} = g_{2j-1,2j-1}, \qquad j = 1, 2, \ldots, l \qquad (3.3)$$

and

$$g_{2j+1,2j+1} = g_{2j-1,2j-1} \exp\{\alpha_j(h)\}, \qquad j = 1, 2, \ldots, l \qquad (3.4)$$

where $\exp\{\alpha_j(h)\}$ are given by equations (2.1). In particular the elements $i\mathbf{h}_j$ of $\mathcal{H} \cap \mathscr{L}$ are transformed so that

$$\mathbf{S}(i\mathbf{h}_j)\mathbf{S}^{-1} = \begin{cases} -\mathbf{M}_{2j+1,2j+2} + \mathbf{M}_{2j-1,2j}, & j = 1, \ldots, l-1 \\ 2\mathbf{M}_{2l-1,2l}, & j = l \end{cases} \qquad (3.5)$$

where

$$\mathbf{M}_{jk} = \mathbf{e}_{jk} - \mathbf{e}_{kj}$$

It will be noted that the only dependence of **S** on the elements of **g** lies in the factor $\sqrt{\mathbf{g}}$ of (3.1), which is defined to be the diagonal matrix such that

$$(\sqrt{\mathbf{g}})_{jj} = \begin{cases} 1, & \text{if } g_{jj} = 1 \\ i, & \text{if } g_{jj} = -1 \end{cases}$$

$\mathbf{T}$ is the matrix that performs the corresponding similarity transformation for the compact case $r = 0$ (Cornwell, 1971a, Appendix B). The remarkable aspect of the theorem is that $\mathbf{S}$ can be cast in such a simple form for *all* cases.

Of course it is possible to rearrange the diagonal elements of $\mathbf{g}$ so as to bring all the elements of the same sign together by a further similarity transformation. For example a further similarity transformation with the matrix $\mathbf{U}$ (where $U_{ts} = U_{st} = 1$, $U_{jj} = 1$ for $j = 1, \ldots, 2l + 1$, $j \neq s, t$, and all other elements zero), which interchange rows s and t and columns s and t, will correspond to interchanging g_{ss} and g_{tt}.

Before outlining the proof of the theorem, two examples will be given for $l = 2$, for which

$$\mathbf{T} = \begin{pmatrix} 0 & i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & i & 0 & -i \\ 0 & 0 & 1 & 0 & 1 \\ \sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}$$

For $r = 1$, i.e. for a mapping onto $so(3, 2)$, equations (2.1), (3.3), and (3.4) imply that the diagonal elements of g are such that $g_{55} = -g_{44} = -g_{33} = g_{22} = g_{11}$. For $r = 2$, i.e. for a mapping onto $so(4, 1)$, equations (2.1), (3.3), and (3.4) imply that $-g_{55} = g_{44} = g_{33} = g_{22} = g_{11}$.

In proving the theorem consider first the compact case in which $r = 0$, $g_{jj} = 1$ for $j = 1, \ldots, 2l + 1$, and $\mathbf{S} = \mathbf{T}$. As implied in Section 2, a general element of $\mathscr{L}_c$ has the form

$$\sum_{j=1}^{l} i\lambda_j \mathbf{h}_j + \sum_{\alpha} \{\tau_\alpha \mathbf{e}_\alpha + \tau_\alpha^* \mathbf{e}_{-\alpha}\}$$

where the λ_j are real and the τ_α are complex. The requirement that $\mathbf{a}$ be real implies that:

$$\mathbf{T}(i\mathbf{h}_j)\mathbf{T}^{-1} \qquad \text{must be real, for } j = 1, 2, \ldots, l \tag{3.6}$$

and, as all the matrices $\mathbf{e}_\alpha$, $\mathbf{e}_{-\alpha}$ are real:

$$\mathbf{e}_\alpha = (\mathbf{T}^{-1}\mathbf{T}^*)\mathbf{e}_{-\alpha}(\mathbf{T}^{-1}\mathbf{T}^*)^{-1} \tag{3.7}$$

for every root α. Also the requirement that $\mathbf{a}$ satisfies (1.1) (i.e. $\tilde{\mathbf{a}} + \mathbf{a} = \mathbf{0}$ in this case) implies that

$$\tilde{\mathbf{T}}\mathbf{T} = \gamma\mathbf{G} \tag{3.8}$$

where γ is some constant and $\mathbf{G}$ is given by (2.6). The simplest matrix $\mathbf{T}$ satisfying (3.6), (3.7) and (3.8) is that given in (3.2). Moreover this $\mathbf{T}$ has the property that

$$\mathbf{T}(i\mathbf{h}_j)\mathbf{T}^{-1} = \begin{cases} -\mathbf{M}_{2j+1,2j+2} + \mathbf{M}_{2j-1,2j}, & j = 1, \ldots, l-1 \\ 2\mathbf{M}_{2l-1,2l}, & j = l \end{cases} \tag{3.9}$$

Turning now to the non-compact cases for which $r \neq 0$, the tentative assumption will first be made that **S** has the form (3.1). It then has to be verified that **a** is real and satisfies (1.1) for an appropriate ordering of diagonal elements of **g**. (The traceless condition is automatically satisfied.) But **a** satisfies (1.1) while every element of $\mathscr{L}$ satisfies (2.5) if and only if

$$\breve{\mathbf{S}}\mathbf{g}\mathbf{S} = \gamma' \mathbf{G} \tag{3.10}$$

where γ' is some constant. However, as a consequence of (3.8), the form (3.1) satisfies (3.10) automatically.

The reality conditions on **a** are more complicated. Firstly, as $i\mathbf{h}_1, i\mathbf{h}_2, \ldots, i\mathbf{h}_l$ are all basis elements of $\mathscr{L}$ (as well as of $\mathscr{L}_c$), the reality condition implies that $\mathbf{S}(i\mathbf{h}_j)\mathbf{S}^{-1}$ must be real for $j = 1, \ldots, l$. Taken with (3.1) and (3.9), this implies that $(\sqrt{\mathbf{g}})^{-1}\mathbf{M}_{2j-1,2j}(\sqrt{\mathbf{g}})$ must be real for $j = 1, \ldots, l$, which can be ensured by the requirement (3.3) (as then $\sqrt{\mathbf{g}}$ commutes with all $\mathbf{M}_{2j-1,2j}$).

Secondly, if $\exp\{\alpha(h)\} = +1$, $(e_\alpha + e_{-\alpha})$ and $i(e_\alpha - e_{-\alpha})$ are members of $\mathscr{L}$ (as well as of $\mathscr{L}_c$), so that, as in (3.7), the reality condition implies that

$$\mathbf{e}_\alpha = (\mathbf{S}^{-1}\mathbf{S}^*)\mathbf{e}_{-\alpha}(\mathbf{S}^{-1}\mathbf{S}^*)^{-1} \tag{3.11}$$

Similarly, if $\exp\{\alpha(h)\} = -1$, as $i(e_\alpha + e_{-\alpha})$ and $(e_\alpha - e_{-\alpha})$ are then basis elements of $\mathscr{L}$, (3.11) is replaced by

$$\mathbf{e}_\alpha = -(\mathbf{S}^{-1}\mathbf{S}^*)\mathbf{e}_{-\alpha}(\mathbf{S}^{-1}\mathbf{S}^*)^{-1}$$

Both cases can be taken together in the requirement that

$$\mathbf{e}_\alpha = \exp\{\alpha(h)\}(\mathbf{S}^{-1}\mathbf{S}^*)\mathbf{e}_{-\alpha}(\mathbf{S}^{-1}\mathbf{S}^*)^{-1}$$

With the form (3.1), as $(\sqrt{\mathbf{g}})^{-1} = (\sqrt{\mathbf{g}})^*$, on using (3.7) this requirement becomes that

$$\mathbf{e}_{-\alpha} = \exp\{\alpha(h)\}(\mathbf{T}^{-1}\mathbf{g}\mathbf{T})\mathbf{e}_{-\alpha}(\mathbf{T}^{-1}\mathbf{g}\mathbf{T})^{-1} \tag{3.12}$$

for every root α. However, with **T** given by (3.2), and with (3.3) satisfied,

$$\mathbf{T}^{-1}\mathbf{g}\mathbf{T} = \operatorname{diag}\{g_{2l+1,2l+1}, g_{11}, g_{33}, \ldots, g_{2l-1,2l-1}, g_{11}, g_{33}, \ldots, g_{2l-1,2l-1}\} \tag{3.13}$$

It is then easily verified that if (3.12) is satisfied for the simple roots, then it is satisfied for all the roots, both positive and negative. As $\mathbf{e}_{-\alpha} = -\breve{\mathbf{e}}_\alpha$, (2.7) and (3.13) show that (3.12) is satisfied for the simple roots if and only if (3.4) is satisfied. Thus all the reality conditions on **a** are met if **g** is chosen in accordance with (3.3) and (3.4).

4. *Similarity Transformation from the Canonical Form to* $so(2l - 2r, 2r)$, $r = 0, 1, \ldots, [\frac{1}{2}l]$

Theorem. Let **b** be an element of the matrix realisation of the canonical form $\mathscr{L}$. Then the similarity transformation to the $so(2l - 2r, 2r)$ Lie algebra (for $r = 0, 1, \ldots, [\frac{1}{2}l]$) is given by

$$\mathbf{a} = \mathbf{S}\mathbf{b}\mathbf{S}^{-1} \tag{4.1}$$

where

$$\mathbf{S} = (\sqrt{\mathbf{g}})\,\mathbf{T} \tag{4.2}$$

and $\mathbf{T}$ is now given by

$$(\mathbf{T})_{jk} = \begin{cases} 1, & j = 2k, \quad k = 1, \ldots, l, \quad \text{and} \\ & j = 2k - 2l, \quad k = l + 1, \ldots, 2l \\ i, & j = 2k - 1, \quad k = 1, \ldots, l \\ -i, & j = 2k - 2l - 1, \quad k = l + 1, \ldots, 2l \\ 0, & \text{all other } j, k \end{cases} \tag{4.3}$$

provided that the diagonal elements of $\mathbf{g}$ of (1.1) are arranged so that

$$g_{2j,2j} = g_{2j-1,2j-1}, \qquad j = 1, \ldots, l \tag{4.4}$$

and

$$g_{2j+2,2j+2} = g_{2j,2j} \exp\{\alpha_j(h)\}, \qquad j = 1, \ldots, l - 1 \tag{4.5}$$

where $\exp\{\alpha_j(h)\}$ are given by equations (2.2). In particular the elements $i\mathbf{h}_j$ of $\mathcal{H} \cap \mathcal{L}$ are transformed so that

$$\mathbf{S}(i\mathbf{h}_j)\mathbf{S}^{-1} = \begin{cases} -\mathbf{M}_{2j+1,2j+2} + \mathbf{M}_{2j-1,2j}, & j = 1, \ldots, l - 1 \\ \mathbf{M}_{2l-3,2l-2} + \mathbf{M}_{2l-1,2l}, & j = l \end{cases} \tag{4.6}$$

Again the only dependence of $\mathbf{S}$ on $\mathbf{g}$ lies in the factor $\sqrt{\mathbf{g}}$. Also $\mathbf{T}$ is again the matrix that performs the similarity transformation for the compact case $r = 0$, so (4.6) is also satisfied with $\mathbf{S}$ replaced by $\mathbf{T}$. As the canonical form $\mathcal{L}$ isomorphic to $so(2l - 2r, 2r)$ is generated by an *inner* involutive automorphism, the method of proof of this theorem is identical to that given for $so(2l + 1 - 2r, 2r)$.

As an example consider the case $l = 3$ and $r = 1$, i.e. the mapping onto $so(4, 2)$. Equations (2.2), (4.4) and (4.5) then imply that $-g_{66} = -g_{55} = -g_{44} = -g_{33} = g_{22} = g_{11}$.

5. *Similarity Transformation from the Canonical Form to* $so(2l - 2r - 1, 2r + 1), r = 0, 1, \ldots, [\frac{1}{2}l]$

Theorem. Let $\mathbf{b}$ be an element of the matrix realisation of the canonical form $\mathcal{L}$. Then the similarity transformation to the $so(2l - 2r - 1, 2r + 1)$ Lie algebra for $r = 0, 1, \ldots, [\frac{1}{2}l]$, is given by (4.1), where $\mathbf{S}$ is again given by (4.2) and $\mathbf{T}$ by (4.3), *provided* that the diagonal elements of $\mathbf{g}$ of (1.1) are arranged so that

$$g_{2j,2j} = g_{2j-1,2j-1}, \qquad j = 1, 2, \ldots, l - 1 \tag{5.1}$$

but

$$g_{2l,2l} = -g_{2l-1,2l-1} \tag{5.2}$$

and

$$g_{2j+2,2j+2} = g_{2j,2j} \exp\{\alpha_j(h)\}, \qquad j = 1, 2, \ldots, l \tag{5.3}$$

where $\exp\{\alpha_j(h)\}$ are given by equations (2.3) or (2.4). In particular the elements of $\mathcal{H} \cap \mathcal{L}$ are transformed so that

$$\left.\begin{aligned} \mathbf{S}(i\mathbf{h}_j)\mathbf{S}^{-1} &= -\mathbf{M}_{2j+1,2j+2} + \mathbf{M}_{2j-1,2j}, \qquad j = 1, \ldots, l-2 \\ \mathbf{S}\{i(\mathbf{h}_{l-1} + \mathbf{h}_l)\}\mathbf{S}^{-1} &= 2\mathbf{M}_{2l-3,2l-2} \\ \mathbf{S}\{\mathbf{h}_{l-1} - h_l\}\mathbf{S}^{-1} &= 2g_{2l,2l}\mathbf{N}_{2l-1,2l} \end{aligned}\right\}$$

where

$$\mathbf{N}_{jk} = \mathbf{e}_{jk} + \mathbf{e}_{kj}$$

As an example consider the case $l = 2$ with $r = 0$, i.e., the mapping onto $so(3, 1)$. Equations (2.4), (5.1), (5.2), and (5.3) then imply that $-g_{44} = g_{33} = g_{22} = g_{11}$, and $\mathbf{S}\{i(\mathbf{h}_1 + \mathbf{h}_2)\}\mathbf{S}^{-1} = 2\mathbf{M}_{12}$, $\mathbf{S}\{\mathbf{h}_1 - \mathbf{h}_2\}\mathbf{S}^{-1} = -2g_{11}\mathbf{N}_{34}$.

The canonical form $\mathcal{L}$ isomorphic to $so(2l - 2r - 1, 2r + 1)$ is generated by an *outer* involutive automorphism, so that the method of proof given in Section 3 has to be modified. However, it is still true that with the form (4.2) for $\mathbf{S}$, the condition (1.1) is automatically satisfied, so that it is only necessary to consider the reality conditions on $\mathbf{a}$.

As before, for the basis vectors $i\mathbf{h}_j, j = 1, \ldots, l-2$, and $i(\mathbf{h}_{l-1} + \mathbf{h}_l)$ the reality condition implies that $(\sqrt{\mathbf{g}})^{-1}\mathbf{M}_{2j-1,2j}(\sqrt{\mathbf{g}})$ must be real for $j = 1, 2, \ldots, l-1$, which can be ensured by the requirement (5.1). However, for the basis element $(\mathbf{h}_{l-1} - \mathbf{h}_l)$ (as (4.6) is also satisfied by $\mathbf{S}$ replaced by $\mathbf{T}$), $\mathbf{S}(\mathbf{h}_{l-1} - \mathbf{h}_l)\mathbf{S}^{-1} = (\sqrt{\mathbf{g}})^{-1} 2i\mathbf{M}_{2l-1,2l}(\sqrt{\mathbf{g}})$, which is real if and only if $g_{2l-1,2l-1} = -g_{2l,2l}$, and when this is so $\mathbf{S}(\mathbf{h}_{l-1} - \mathbf{h}_l)\mathbf{S}^{-1} = 2g_{2l,2l}\mathbf{N}_{2l-1,2l}$. For the other basis vectors it is easily shown that the generalisation of the condition (3.12) is

$$\mathbf{e}_\alpha = \exp\{\alpha(h)\}(\mathbf{T}^{-1}\mathbf{g}\mathbf{T})\{Z_0\mathbf{e}_\alpha\}(\mathbf{T}^{-1}\mathbf{g}\mathbf{T})^{-1} \tag{5.4}$$

where Z_0 is the automorphism described in Section 2. It is easily verified that if (5.4) is satisfied for the simple roots, then it is satisfied for all the positive roots, and for the negative roots as well if

$$(\mathbf{T}^{-1}\mathbf{g}\mathbf{T})_{jk} = (\mathbf{T}^{-1}\mathbf{g}\mathbf{T})_{kj}, \qquad j, k = 1, \ldots, 2l \tag{5.5}$$

But, with (4.3), (5.1) and (5.2),

$$\left.\begin{aligned} (\mathbf{T}^{-1}\mathbf{g}\mathbf{T})_{jj} &= g_{2j+1,2j+1}, \qquad j = 1, \ldots, l-1 \\ (\mathbf{T}^{-1}\mathbf{g}\mathbf{T})_{jj} &= g_{2j-2l-1,2j-2l-1}, \qquad j = l+1, \ldots, 2l-1 \\ (\mathbf{T}^{-1}\mathbf{g}\mathbf{T})_{l,2l} &= (\mathbf{T}^{-1}\mathbf{g}\mathbf{T})_{2l,l} = g_{2l,2l} \end{aligned}\right\} \tag{5.6}$$

with all other elements zero, so that (5.5) is satisfied. It is therefore only necessary to test (5.4) for the simple roots. For $\alpha_1, \alpha_2, \ldots, \alpha_{l-2}$, as $Z_0e_j = e_j$, the condition (5.4) is equivalent to (3.12) and again gives (5.3) (for $j = 1, 2, \ldots, l-2$). For α_{l-1} and α_l, as $Z_0e_{l-1} = e_l$ and $Z_0e_l = e_{l-1}$, by (2.9) and (5.6) the condition (5.4) is satisfied if $g_{2l,2l} = g_{2l-2,2l-2} \exp\{\alpha_l(h)\}$.

6. *Mapping of the Maximal Compact Subalgebra*

The 'natural' maximal compact subalgebra $\mathscr{K}$ of $\mathscr{L}$ may be defined as the set of elements $b \in \mathscr{L}$ such that $Zb = b$, where Z is the chief involutive automorphism described in Section 2. Of course any automorphism of $\mathscr{L}$ applied to $\mathscr{K}$ gives another maximal compact subalgebra of $\mathscr{L}$ that is isomorphic to $\mathscr{K}$.

The 'natural' maximal compact subalgebra $\mathscr{K}'$ of $so(p,q)$ may be defined as the set of matrices $\mathbf{a} \in so(p,q)$ such that $a_{pq} = 0$ if $g_{pp} \neq g_{qq}$, where $\mathbf{g}$ is the matrix of equation (1.1). Clearly $\mathscr{K}'$ is isomorphic to $so(p) \oplus so(q)$, and again any automorphism of $so(p,q)$ applied to $\mathscr{K}'$ gives another maximal subalgebra of that is isomorphic to $\mathscr{K}'$.

Theorem. The isomorphic mappings of $\mathscr{L}$ onto $so(p,q)$ considered in Sections 3, 4 and 5 each map the *natural* maximal compact subalgebra $\mathscr{K}$ onto the *natural* maximal compact subalgebra $\mathscr{K}'$.

It is obvious that $\mathscr{K}$ must be mapped onto a subalgebra of $so(p,q)$ that is isomorphic to $\mathscr{K}'$. What is remarkable about this theorem is that $\mathscr{K}$ is mapped directly onto $\mathscr{K}'$ itself.

An outline of the proof will be given for the cases in which Z is an inner automorphism. (The case in which Z is an outer automorphism is more difficult. In fact the simplest approach for this case is that of direct verification.) If $Z = \exp(\mathrm{ad}\, h)$ is a chief inner automorphism, the basis elements of $\mathscr{K}$ consist of $ih_j, j = 1, \ldots, l$ together with $(e_\alpha + e_{-\alpha})$ and $i(e_\alpha - e_{-\alpha})$ for roots α such that $\exp\{\alpha(h)\} = +1$. It follows immediately from equations (3.5) and (4.6) (taken with (3.3) and (4.4) respectively) that $ih_j, j = 1, \ldots, l$ are mapped into $\mathscr{K}'$. For $(e_\alpha + e_{-\alpha})$ and $i(e_\alpha - e_{-\alpha})$ with $\exp\{\alpha(h)\} = 1$, as $\sqrt{\mathbf{g}}$ is diagonal and as $\mathbf{T}\mathbf{e}_{-\alpha}\mathbf{T}^{-1} = (\mathbf{T}\mathbf{e}_\alpha\mathbf{T}^{-1})^*$ by (3.7), it has only to be shown that $(\mathbf{T}\mathbf{e}_\alpha\mathbf{T}^{-1})_{pq} = 0$ when $g_{pp} \neq g_{qq}$. In fact it is easier to demonstrate the more general proposition that $(\mathbf{T}\mathbf{e}_\alpha\mathbf{T}^{-1})_{pq} = 0$ when $g_{pp} \neq g_{qq} \exp\{\alpha(h)\}$ for every root α, for if this proposition is true for the simple roots then it is true for all the roots. But equations (3.3), (3.4), (4.4) and (4.5) show that the proposition is true for the simple roots, as direct calculation for B_l gives

$$\mathbf{T}\mathbf{e}_j\mathbf{T}^{-1} = \begin{cases} \frac{1}{2}\{\mathbf{M}_{2j-1,2j+1} + i\mathbf{M}_{2j-1,2j+2} - i\mathbf{M}_{2j,2j+1} \\ \qquad + \mathbf{M}_{2j,2j+2}\}\{2(2l-1)\}^{-1/2}, & j = 1, \ldots, l-1 \\ \frac{1}{2}\{\mathbf{M}_{2l+1,2l} + i\mathbf{M}_{2l+1,2l-1}\}\{2l-1\}^{-1/2}, & j = l \end{cases}$$

and for D_l gives

$$\mathbf{T}\mathbf{e}_j\mathbf{T}^{-1} = \begin{cases} \frac{1}{2}\{\mathbf{M}_{2j-1,2j+1} + i\mathbf{M}_{2j-1,2j+2} - i\mathbf{M}_{2j,2j+1} \\ \qquad + \mathbf{M}_{2j,2j+2}\}\{4(2l-1)\}^{-1/2}, & j = 1, \ldots, l-1 \\ \frac{1}{2}\{-\mathbf{M}_{2l-3,2l-1} + i\mathbf{M}_{2l-3,2l} + i\mathbf{M}_{2l-2,2l-1} \\ \qquad + \mathbf{M}_{2l-2,2l}\}\{4(l-1)\}^{-1/2}, & j = l. \end{cases}$$

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